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Certain Properties of a Pair of Secondary Zeta-Functions

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1. INTRODUCTION

In a recent paper [2], we defined the "secondary zeta-functions" as a pair of functions derived from the Dirichlet series involving the powers of primes in one case and the non-trivial zeros of the Riemann zeta-function in the other and showed that the two secondary zeta-functions are related by a formula analogous to the functional equation of the Riemann zeta-function itself. This paper deals with a few properties of a pair of secondary zeta-functions.

For convenience of notation we make the convention throughout that $\frac{1}{2} + i\gamma$ shall run through the complex zeros of $\zeta(s)$, the Riemann zeta-function. Thus some values of γ will be complex if the Riemann hypothesis is false. Furthermore, if multiple zeros of $\zeta(s)$ exist, then the terms in sums over γ shall occur in their appropriate order of multiplicity.

The secondary zeta-functions $Z_p(s)$ and $Z_\gamma(s)$, whose properties we are going to investigate, are defined by:

$$Z_p(s) = \lim_{T\to\infty} \left\{ \sum_{0<m\log p<T} \frac{\log p}{p^{\frac{1}{2}m}} (m \log p)^{-s} - \int_0^T e^{\frac{1}{2}u} u^{-s}\, du \right\},$$

where p runs through the prime numbers and m through the positive integers, and

$$Z_\gamma(s) = \sum_{\mathrm{Re}(\gamma)>0} \gamma^{-s}.$$

The Dirichlet series involved in the definition of the function $Z_p(s)$ does not converge in the ordinary sense, but we can overcome this difficulty by using the notion of modified Abel summability which was used by Atkinson [1] to deal with certain divergent Dirichlet series in the theory of partitions.

The paper consists of four sections. In the section following this introduction, the poles of the function $Z_p(s)$ have been investigated. The behaviour of the function $Z_\gamma(s)$ near $s = 1$ have been examined in the next section. Some particular values of the functions $Z_p(s)$ and $Z_\gamma(s)$ have been calculated in the last section.

2. The Poles of the Function $Z_p(s)$

We have the Equation ([3], p. 58):

$$\frac{\zeta'}{\zeta}(s) = b - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}s + 1) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{2.1}$$

where $b = \log 2\pi - \frac{1}{2}C - 1$, C being the Euler's constant, and $\rho = \frac{1}{2} + i\gamma$ runs through the nontrivial zeros of the function $\zeta(s)$.

Putting $s = \frac{1}{2} + t$ in (2.1), we obtain:

$$\frac{\zeta'}{\zeta}(\tfrac{1}{2} + t) = b - \frac{1}{t - \frac{1}{2}} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{5}{4} + \tfrac{1}{2}t) + \sum_{\mathrm{Re}(\gamma)>0}\left(\frac{2t}{t^2+\gamma^2} + \frac{1}{\frac{1}{4}+\gamma^2}\right). \tag{2.2}$$

We also have the result ([7], p. 247):

$$\frac{\Gamma'}{\Gamma}(1 + z) = -C + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right). \tag{2.3}$$

Writing $\frac{1}{4} + \frac{1}{2}t$ for z in (2.3), we obtain:

$$\frac{\Gamma'}{\Gamma}(\tfrac{5}{4} + \tfrac{1}{2}t) = -C + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n + \frac{1}{4} + \frac{1}{2}t}\right). \tag{2.4}$$

Equations (2.2) and (2.4) now give:

$$-\frac{\zeta'}{\zeta}(\tfrac{1}{2} + t) - \frac{1}{t - \frac{1}{2}} = -b - \tfrac{1}{2}C + \tfrac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n + \frac{1}{4} + \frac{1}{2}t}\right)$$
$$- 2t\sum_{\mathrm{Re}(\gamma)>0}\frac{1}{t^2+\gamma^2} - \sum_{\mathrm{Re}(\gamma)>0}\frac{1}{\frac{1}{4}+\gamma^2}.$$

Therefore, if $f(t)$ represents the derivative with respect to t of the function on the left of the equation above, then

$$f(t) = \sum_{n=1}^{\infty}\frac{1}{(t+a)^2} + 2\sum_{\mathrm{Re}(\gamma)>0}\frac{t^2-\gamma^2}{(t^2+\gamma^2)^2}, \tag{2.5}$$

where

$$a = 2n + \tfrac{1}{2}.$$

The function $f(t)$, thus defined, is regular at $t = \frac{1}{2}$ and consequently regular for all $t \geqslant 0$. As $t \to \infty$, it is of the order $0(t^{-2})$ and therefore, the function $F(s)$, the Mellin transform of $f(t)$, given by

$$F(s) = \int_0^\infty f(t)\, t^{s-1}\, dt \tag{2.6}$$

converges for $0 < \mathrm{Re}(s) < 2$. It is, however, possible to get an analytic continuation for $F(s)$.

Now, for $0 < \mathrm{Re}(s) < 2$, we have:

$$\int_0^\infty \frac{t^{s-1}}{(t+a)^2}\, dt = \frac{\pi(1-s)\, a^{s-2}}{\sin s\pi}, \tag{2.7}$$

and

$$\int_0^\infty \frac{t^2 - \gamma^2}{(t^2+\gamma^2)^2}\, t^{s-1}\, dt = \frac{\pi(s-1)\, \gamma^{s-2}}{2 \sin \frac{1}{2}s\pi}. \tag{2.8}$$

Therefore, using (2.5) in (2.6), we have, with the help of (2.7) and (2.8):

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} \frac{\pi(1-s)\, a^{s-2}}{\sin s\pi} + \sum_{\mathrm{Re}(\gamma)>0} \frac{\pi(s-1)\, \gamma^{s-2}}{\sin \frac{1}{2}s\pi} \\ &= \frac{\pi(1-s)}{\sin s\pi} \chi(2-s) - \frac{\pi(1-s)}{\sin \frac{1}{2}s\pi} Z_\gamma(2-s), \qquad (\mathrm{Re}(s) < 1), \end{aligned} \tag{2.9}$$

where

$$\chi(s) = 2^s \sum_{n=1}^{\infty} (4n+1)^{-s}, \qquad (\mathrm{Re}(s) > 1) \tag{2.10}$$

and

$$Z_\gamma(s) = \sum_{\mathrm{Re}(\gamma)>0} \gamma^{-s}, \qquad (\mathrm{Re}(s) > 1).$$

Now, the functional equation connecting $Z_p(s)$ and $Z_\gamma(s)$ is given by [2]:

$$\begin{aligned} &Z_p(s) - \Gamma(1-s)\{2^{1-s} - 2^{-s}\eta(1-s) - (2^{-s} - \tfrac{1}{2})\, \zeta(1-s)\} \\ &\quad = -2\Gamma(1-s) \sin \tfrac{1}{2}s\pi Z_\gamma(1-s), \end{aligned} \tag{2.11}$$

where

$$\eta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s}, \qquad (\mathrm{Re}(s) > 1). \tag{2.12}$$

We may write

$$\eta(s) = \sum_{n=0}^{\infty} (4n + 1)^{-s} - \sum_{n=0}^{\infty} (4n + 3)^{-s},$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty} (4n + 1)^{-s} &= \sum_{n=0}^{\infty} (4n + 1)^{-s} - 1 \\ &= \eta(s) + \sum_{n=0}^{\infty} (4n + 3)^{-s} - 1 \\ &= \tfrac{1}{2}\eta(s) + \tfrac{1}{2} \sum_{n=0}^{\infty} (2n + 1)^{-s} - 1 \\ &= \tfrac{1}{2}\eta(s) + \tfrac{1}{2}(1 - 2^{-s})\, \zeta(s) - 1,\end{aligned}$$

that is,

$$\tfrac{1}{2}\eta(s) + \tfrac{1}{2}(1 - 2^{-s})\, \zeta(s) = 1 + 2^{-s}\chi(s), \qquad (\mathrm{Re}(s) > 1). \tag{2.13}$$

Replacing s by $1 - s$, this can be written as:

$$2^{-s}\eta(1 - s) + (2^{-s} - \tfrac{1}{2})\, \zeta(1 - s) = 2^{1-s} + \chi(1 - s), \qquad (\mathrm{Re}(s) < 0). \tag{2.14}$$

Using (2.14), we can now write the functional Eq. (2.11) as:

$$Z_p(s) = - \Gamma(1 - s)\, \chi(1 - s) - 2\Gamma(1 - s) \sin \tfrac{1}{2}s\pi Z_p(2 - s), \qquad (\mathrm{Re}(s) < 0). \tag{2.15}$$

Hence, with the help of (2.9), we can write:

$$\begin{aligned}Z_p(s - 1) &= - \frac{\Gamma(2 - s) \sin s\pi}{\pi(1 - s)} F(s), \qquad (\mathrm{Re}(s) < 1) \\ &= - \frac{F(s)}{\Gamma(s)} \\ &= - \frac{1}{\Gamma(s)} \int_0^{\infty} f(t)\, t^{s-1}\, dt, \qquad (1 < \mathrm{Re}(s) < 2),\end{aligned} \tag{2.16}$$

using the analytic continuation of $F(s)$.

But, by definition,

$$f(t) = - \frac{d}{dt} \left\{ \frac{\zeta'}{\zeta} (\tfrac{1}{2} + t) \right\} + \frac{1}{(t - \frac{1}{2})^2}$$

Writing this for $f(t)$ and integrating by parts, (2.16) becomes:

$$- \frac{1}{\Gamma(s - 1)} \int_0^{\infty} \left\{ \frac{\zeta'}{\zeta} (\tfrac{1}{2} + t) + \frac{1}{t - \frac{1}{2}} \right\} t^{s-2}\, dt,$$

since the integrated term vanishes for $1 < \mathrm{Re}(s) < 2$.

Hence,

$$Z_p(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left\{ -\frac{\zeta'}{\zeta}(\tfrac{1}{2} + t) - \frac{1}{t - \frac{1}{2}} \right\} t^{s-1}\, dt, \qquad (0 < \operatorname{Re}(s) < 1).$$

To get the analytic continuation for $Z_p(s)$, we write:

$$\begin{aligned} Z_p(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \left\{ -\frac{\zeta'}{\zeta}(\tfrac{1}{2} + t) - \frac{1}{t - \frac{1}{2}} + \frac{1 - e^{1/2-t}}{t - \frac{1}{2}} \right\} t^{s-1}\, dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^\infty \frac{1 - e^{1/2-t}}{\frac{1}{2} - t}\, t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \left\{ -\frac{\zeta'}{\zeta}(\tfrac{1}{2} + t) - \frac{e^{1/2-t}}{t - \frac{1}{2}} \right\} t^{s-1}\, dt - \int_0^1 e^{\frac{1}{2}u} u^{-s}\, du, \end{aligned}$$

since

$$\begin{aligned} \int_0^\infty \frac{1 - e^{1/2-t}}{\frac{1}{2} - t}\, t^{s-1}\, dt &= -\int_0^\infty t^{s-1}\, dt \int_0^1 e^{(1/2-t)u}\, du \\ &= -\Gamma(s) \int_0^1 e^{\frac{1}{2}u} u^{-s}\, du, \qquad (0 < \operatorname{Re}(s) < 1). \end{aligned}$$

We can now say that

$$Z_p(s) + \int_0^1 e^{\frac{1}{2}u} u^{-s}\, du$$

has a single valued analytic continuation for $\operatorname{Re}(s) > 0$ with no poles.

For $\operatorname{Re}(s) < 1$, we have

$$\int_0^1 e^{\frac{1}{2}u} u^{-s}\, du = \sum_{n=0}^\infty \frac{(\frac{1}{2})^n}{n!} \int_0^1 u^{n-s}\, du = \sum_{n=0}^\infty \frac{1}{2^n n!\,(n + 1 - s)}\,.$$

This shows that $\int_0^1 e^{\frac{1}{2}u} u^{-s}\, du$ has a continuation with simple poles of residues

$$-\frac{1}{2^{m-1}(m-1)!} \quad \text{at} \quad s = m, \qquad (m = 1, 2, 3, \ldots).$$

Therefore, the function $Z_p(s)$ has simple poles of residues

$$\frac{1}{2^{m-1}(m-1)!} \quad \text{at} \quad s = m, \qquad (m = 1, 2, 3, \ldots).$$

3. The Behaviour of the Function $Z_\gamma(s)$ near $s = 1$

To begin with we assume the Riemann hypothesis to be correct and let $N(x)$ denote the number of zeros of the function $\zeta(s)$ in $0 < \operatorname{Im}(s) \leqslant x$.

Consider the function

$$T(s) = \int_0^\infty \left\{ N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right) \right\} x^{s-2}\, dx, \tag{3.1}$$

It is known ([6], p. 181) that the function

$$S(x) = N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right)$$

is of order $0(\log x)$ as $x \to \infty$. Hence, the integral (3.1) exists when x is infinite if $\mathrm{Re}(s) < 1$. As $x \to 0$, the integrand is of order $0(x^{s-1} \log x)$ and hence the integral exists at $x = 0$ if $\mathrm{Re}(s) > 0$. Therefore, the integral (3.1) converges for $0 < \mathrm{Re}(s) < 1$.

We write:

$$T(s) = \int_1^\infty \left\{ N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right) \right\} x^{s-2}\, dx$$
$$- \frac{1}{2\pi} \int_0^1 \left(x^{s-1} \log \frac{x}{2\pi} - x^{s-1} \right) dx.$$

The first integral on the right converges for $\mathrm{Re}(s) < 1$ and for $\mathrm{Re}(s) > 0$, the second integral is equal to

$$- \frac{1}{2\pi} \left\{ \frac{1 + \log 2\pi}{s} + \frac{1}{s^2} \right\}.$$

Therefore, we have:

$$T(s) = \int_1^\infty \left\{ N(x) - \left(\frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} \right) \right\} x^{s-2}\, dx$$
$$+ \frac{1}{2\pi s^2} + \frac{1 + \log 2\pi}{2\pi s}, \qquad (\mathrm{Re}(s) < 1). \tag{3.2}$$

This will give us the analytic continuation of $T(s)$ for $\mathrm{Re}(s) < 0$. We can write:

$$T(s) = \int_1^\infty N(x)\, x^{s-2}\, dx - \frac{1}{2\pi} \int_1^\infty \left(x \log \frac{x}{2\pi} - x \right) x^{s-2}\, dx$$
$$+ \frac{1}{2\pi s^2} + \frac{1 + \log 2\pi}{2\pi s}, \qquad (\mathrm{Re}(s) < 0). \tag{3.3}$$

Integrating by parts, the second integral in (3.3) can be shown to be equal to

$$\frac{1}{2\pi} \left\{ \frac{1}{s^2} + \frac{1 + \log 2\pi}{s} \right\}.$$

Therefore, Eq. (3.3) becomes:

$$T(s) = \int_1^\infty N(x)\, x^{s-2}\, dx, \qquad (\mathrm{Re}(s) < 0)$$
$$= \frac{1}{1-s}\int_1^\infty x^{s-1}\, dN(x).$$

Now, without the Riemann hypothesis, we have:

$$N(x) = \sum_{0<\mathrm{Re}(\gamma)\leqslant x} 1$$

and so

$$T(s) = \frac{1}{1-s}\sum_\gamma \frac{1}{\{\mathrm{Re}(\gamma)\}^{1-s}} \tag{3.4}$$

Returning now to Equation (3.2), we obtain, using (3.4):

$$\frac{1}{1-s}\sum_\gamma \frac{1}{\{\mathrm{Re}(\gamma)\}^{1-s}} = \frac{1}{2\pi s^2} + \frac{1+\log 2\pi}{2\pi s} + \int_1^\infty S(x)\, x^{s-2}\, dx, \quad (\mathrm{Re}(s) < 1).$$

Replacing s by $1 - s$, we have:

$$\frac{1}{s}\sum_\gamma \frac{1}{\{\mathrm{Re}(\gamma)\}^{s}} = \frac{1}{2\pi(1-s)^2} + \frac{1+\log 2\pi}{2\pi(1-s)} + \int_1^\infty S(x)\, x^{-s-1}\, dx, \quad (\mathrm{Re}(s) > 0).$$

As $x \to \infty$, $S(x) = 0(\log x)$ and so the last integral is regular at $s = 1$. We can therefore write, near $s = 1$:

$$\sum_\gamma \frac{1}{\{\mathrm{Re}(\gamma)\}^{s}} = \frac{1}{2\pi(s-1)^2} - \frac{\log 2\pi}{2\pi(s-1)} + 0(1) \tag{3.5}$$

But since $|\,\mathrm{Im}(\gamma)| < \frac{1}{2}$, we have:

$$\left|\frac{1}{\gamma^s} - \frac{1}{\{\mathrm{Re}(\gamma)\}^{s}}\right| = 0\left(\frac{1}{\gamma^{s+1}}\right)$$

and so

$$\sum_\gamma \left\{\frac{1}{\gamma^s} - \frac{1}{\{\mathrm{Re}(\gamma)\}^{s}}\right\}$$

would converge at $s = 1$ (as in $\sum_\gamma 1/\gamma^2$). This means that the difference is regular at $s = 1$. Hence, $Z_\gamma(s)$ behaves as the expression on the right-hand side of (3.5) near $s = 1$ and we may write:

$$Z_\gamma(s) = \frac{1}{2\pi(s-1)^2} - \frac{\log 2\pi}{2\pi(s-1)} + 0(1). \tag{3.6}$$

We can now conclude that the function $Z_y(s)$ has a double pole at $s = 1$ with residue $-(\log 2\pi)/2\pi$ and principal part

$$\frac{1}{2\pi(s-1)^2} - \frac{\log 2\pi}{2\pi(s-1)}.$$

4. Some Particular Values of the Functions $Z_p(s)$ and $Z_y(s)$

From Section 2 of this paper we have the functional equations [Eqs. (2.15) and (2.13)]:

$$Z_p(s) = -\Gamma(1-s)\,\chi(1-s) - 2\Gamma(1-s)\sin \tfrac{1}{2}s\pi Z_y(1-s) \tag{4.1}$$

and

$$\chi(s) = 2^{s-1}\eta(s) + (2^{s-1} - \tfrac{1}{2})\,\zeta(s) - 2^s, \tag{4.2}$$

where $\chi(s)$ and $\eta(s)$ are given by (2.10) and (2.12) respectively.

The function $\eta(s)$ also satisfies a functional equation which can be derived from the general summation formula [5, p. 66]:

$$\sqrt{\alpha}\,\{f(\alpha) - f(3\alpha) + \cdots\} = \sqrt{\beta}\,\{F_s(\beta) - F_s(3\beta) + \cdots\}, \tag{4.3}$$

where $F_s(x)$ is the Fourier sine transform of $f(x)$ and $\alpha\beta = \pi/2$. If we let

$$f(x) = x^{-s} \qquad (0 < \mathrm{Re}(s) < 1)$$

then

$$\begin{aligned} F_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty t^{-s}\sin xt\,dt \\ &= \sqrt{\frac{2}{\pi}}\,x^{s-1}\,\mathrm{Im}\left(\int_0^\infty e^{iu}u^{-s}\,du\right) \\ &= \sqrt{\frac{2}{\pi}}\,x^{s-1}\Gamma(1-s)\cos \tfrac{1}{2}s\pi. \end{aligned}$$

With these $f(x)$ and $F_s(x)$, the conditions of the summation formula are satisfied and the rightside of (4.3) becomes:

$$\sqrt{2\beta/\pi}\,\Gamma(1-s)\cos \tfrac{1}{2}s\pi\{\beta^{s-1} - (3\beta)^{s-1} + \cdots\},$$

Hence, using $\alpha\beta = \pi/2$, it follows from (4.3) that the functional equation for $\eta(s)$ can be written as:

$$\eta(s) = (\pi/2)^{s-1}\cos \tfrac{1}{2}s\pi\Gamma(1-s)\,\eta(1-s). \tag{4.4}$$

This gives:

$$\eta(0) = \tfrac{1}{2},$$

since

$$\eta(1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4};$$

$$\eta(1 - 2m) = 0 \quad \text{and} \quad \eta(-2m) = \tfrac{1}{2} E_{2m},$$

since [4, p. 240]

$$\eta(2m + 1) = (-1)^m \frac{E_{2m}}{2^{2m+2}(2m)!} \pi^{2m+1}, \tag{4.5}$$

where E_{2m} is an Eulerian number.

We also have the following results [6, p. 19]:

$$\zeta(0) = -\tfrac{1}{2}, \qquad \zeta(-2m) = 0$$

and

$$\zeta(1 - 2m) = \frac{(-1)^m B_{2m}}{2m},$$

where B_{2m} is a Bernoulli number.

Hence, we obtain from (4.2):

$$\chi(0) = -\tfrac{3}{4},$$

$$\chi(1 - 2m) = (-1)^m(2^{-2m} - \tfrac{1}{2}) \frac{B_{2m}}{2m} - 2^{1-2m}$$

and

$$\chi(-2m) = 2^{-2m}(\tfrac{1}{4}E_{2m} - 1).$$

We aslo know from Section 2 of this paper that the function $Z_p(s)$ has simple poles at $s = m$, $(m = 1, 2, 3,...)$ with residues

$$\frac{1}{2^{m-1}(m-1)!}, \qquad (m = 1, 2, 3,...).$$

Since $Z_p(s)$ has a simple pole at $s = 1$ with residue 1, we must have

$$\lim_{s\to 1} (s - 1) Z_p(s) = 1.$$

Hence, multiplying the functional Equation (4.1) throughout by $s - 1$ and taking limits of both sides $s \to 1$, we obtain:

$$\begin{aligned} 1 &= \{\chi(0) + 2Z_\gamma(0)\} \lim_{s\to 1} \{1 - s) \Gamma(1 - s)\} \\ &= \chi(0) + 2Z_\gamma(0), \end{aligned}$$

whence

$$Z_\gamma(0) = \tfrac{7}{8}.$$

In the same way we can calculate $Z_\gamma(-2m)$. Considering the residue of $Z_p(s)$ at $s = 2m + 1$, we get:

$$\lim_{s \to (2m+1)} (s - 2m - 1)\, Z_p(s) = \frac{1}{2^{2m}(2m)!}.$$

Therefore, multiplying the functional Eq. (4.1) throughout by $s - 2m - 1$ and taking limits of both sides as $s \to 2m + 1$, we obtain:

$$\frac{1}{2^{2m}(2m)!} = \{\chi(-2m) + 2(-1)^m Z_\gamma(-2m)\} \lim_{s \to 2m+1} \{(2m + 1 - s)\, \Gamma(1 - s)\}. \tag{4.6}$$

But

$$\lim_{s \to 2m+1} \{(2m + 1 - s)\, \Gamma(1 - s)\} = \lim_{s \to 2m+1} \frac{(2m + 1 - s)\pi}{\Gamma(s) \sin s\pi}$$

$$= \frac{1}{\Gamma(2m + 1)} = \frac{1}{(2m)!},$$

and so Eq. (4.6) gives:

$$Z_\gamma(-2m) = (-1)^m \frac{8 - E_{2m}}{2^{2m+3}}.$$

In particular, using $E_2 = -1$, we get:

$$Z_\gamma(-2) = -\frac{9}{32}.$$

To find $Z_p(0)$, we again consider the functional Eq. (4.1) and take limits of both sides as $s \to 0$. We obtain:

$$Z_p(0) = -\lim_{s \to 0} \{\chi(1 - s) + 2 \sin \tfrac{1}{2} s\pi Z_\gamma(1 - s)\}, \tag{4.7}$$

where

$$\chi(1 - s) = 2^{-s}\eta(1 - s) + (2^{-s} - \tfrac{1}{2})\, \zeta(1 - s) - 2^{1-s}.$$

Now, $\eta(1) = \pi/4$ and near $s = 0$, the function $\zeta(1 - s)$ can be written as ([6], p. 16):

$$\zeta(1 - s) = -\frac{1}{s} + C + 0(s),$$

and

$$2^{-s} = 1 - s \log 2 + 0(s^2).$$

Also, from the previous section we know that near $s = 0$, the function $Z_\gamma(1 - s)$ can be written as (Eq. 3.6):

$$Z_\gamma(1 - s) = \frac{1}{2\pi s^2} + \frac{\log 2\pi}{2\pi s} + 0(1).$$

Hence, using the usual series expansion for $\sin \frac{1}{2} s\pi$, we can write (4.7) as:

$$2 - \frac{\pi}{4} - \lim_{s \to 0} \Big[\{ \tfrac{1}{2} - s \log 2 + 0(s^2) \} \Big\{ -\frac{1}{s} + C + 0(s) \Big\}$$

$$+ \{ s\pi + 0(s^3) \} \Big\{ \frac{1}{2\pi s^2} + \frac{\log 2\pi}{2\pi s} + 0(1) \Big\} \Big]$$

$$= 2 - \Big(\frac{\pi}{4} + \frac{1}{2} C + \frac{1}{2} \log 8\pi \Big).$$

But if we put $s = \frac{1}{2}$ in the known functional equation:

$$\frac{\zeta'}{\zeta}(s) = \log 2\pi + \frac{1}{2} \pi \cot \frac{1}{2} s\pi - \frac{\Gamma'}{\Gamma}(1 - s) - \frac{\zeta'}{\zeta}(1 - s),$$

we find

$$\frac{\zeta'}{\zeta}\Big(\frac{1}{2}\Big) = \frac{\pi}{4} + \frac{1}{2} C + \frac{1}{2} \log 8\pi.$$

Therefore

$$Z_p(0) = 2 - \frac{\zeta'}{\zeta}\Big(\frac{1}{2}\Big).$$

To find $Z_p(-2m)$, we observe that the function $Z_\gamma(s)$ is regular at $s = 2m + 1$, $(m = 1, 2, 3,...)$ and therefore the functional Eq. (4.1) should give us values of $Z_p(-2m)$, $(m = 1, 2, 3,...)$. Putting $s = -2m$ in the aforesaid functional equation, we obtain:

$$Z_p(-2m) = -(2m)! \{ 2^{2m} \eta(2m + 1) + (2^{2m} - \tfrac{1}{2}) \zeta(2m + 1) - 2^{2m+1} \}$$
$$= (2m)! \{ 2^{2m+1} - (2^m - \tfrac{1}{2}) \zeta(2m + 1) \} - \tfrac{1}{4}(-1)^m \pi^{2m+1} E_{2m},$$

using (4.5).

In particular, using $\eta(3) = \pi^3/32$, we get:

$$Z_p(-2) = 16 - \frac{\pi^3}{4} - 7\zeta(3).$$

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